Ergodic Theory - Week 3

Course Instructor: Florian K. Richter
Teaching assistant: Konstantinos Tsinas

1 Uniform Distribution of Sequences

P1. Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system and let $A \in \mathcal{A}$ be a set of positive measure. Prove Khintchine's theorem: for any $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} \colon \mu(A \cap T^{-n}A) \geq (\mu(A))^2 - \varepsilon\}$$

has bounded gaps.

Let $f = \mathbb{1}_A$ be the indicator function of our set A and let $\varepsilon > 0$. Applying the uniform mean ergodic theorem from the last notes, we deduce that

$$\lim_{M-N\to+\infty} \frac{1}{M-N} \sum_{n=N}^{M-1} T^n f = f_{\text{inv}}.$$

where convergence takes place in $L^2(\mu)$. This "strong convergence" implies the following "weak convergence": for any function $g \in L^2(\mu)$, we have

$$\lim_{M-N\to+\infty} \int g \left(\frac{1}{M-N} \sum_{n=N}^{M-1} T^n f \right) d\mu = \int g \cdot f_{inv} \, d\mu,$$

which is a consequence of the mean ergodic theorem (prove it by taking the difference of the two expressions above and use the Cauchy-Schwarz inequality)

We apply this for $g = f = \mathbb{1}_A$ to deduce that

$$\lim_{M-N\to+\infty} \frac{1}{M-N} \sum_{n=N}^{M-1} \int f \cdot T^n f d\mu = \int f \cdot f_{inv} \, d\mu.$$

A simple calculation yields

$$\int f \cdot T^n f \, d\mu = \int \mathbb{1}_A \cdot T^n \mathbb{1}_A \, d\mu = \int \mathbb{1}_{A \cap T^{-n}A} \, d\mu = \mu(A \cap T^{-n}A).$$

In addition, we have

$$\int f \cdot f_{inv} d\mu = \int \overline{f} \cdot f_{inv} d\mu = \int |f_{inv}|^2 \, d\mu + \int \overline{f_{erg}} \cdot f_{inv} d\mu = \int |f_{inv}|^2 d\mu \geq \left| \int f_{inv} \, d\mu \right|^2,$$

where we used the fact that f_{erg} is orthogonal to f_{inv} and the Cauchy-Schwarz inequality in the last step. In addition, we have $\int f_{inv} \, d\mu = \int f \, d\mu = \mu(A)$, since $\int f_{erg} = 0$ (it is orthogonal to the constant function 1).

Putting everything together, we conclude that

$$\lim_{M-N\to+\infty}\frac{1}{M-N}\sum_{n=N}^{M-1}\mu(A\cap T^{-n}A)\geq(\mu(A))^2.$$

Therefore, there exists $K\in\mathbb{N}$, such that for all natural numbers $N<M$ with $N-M>K$, we have

$$\frac{1}{M-N}\sum_{n=N}^{M-1}\mu(A\cap T^{-n}A)\geq(\mu(A))^2-\varepsilon.$$

This shows that the set of n for which $\mu(A\cap T^{-n}A)\geq(\mu(A))^2-\varepsilon$ has gaps bounded by $2K$. Indeed, if that was not the case, one could find $N<M$ with $N-M>2K$, such that all $n\in[N,M]\cap\mathbb{N}$ satisfy $\mu(A\cap T^{-n}A)<(\mu(A))^2-\varepsilon$. This contradicts the fact that the average above is larger than $(\mu(A))^2-\varepsilon$, because if each term of the average is strictly smaller than $(\mu(A))^2-\varepsilon$, then their average cannot exceed $(\mu(A))^2-\varepsilon$.

P2. Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1 if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\{x_n\}^h=\frac{1}{h+1},\forall h\in\mathbb{N}.$$

($\Longrightarrow$) If the sequence is uniformly distributed, then Weyl's criterion applied to the function $f(x)=x^l$ implies that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\{x_n\}^h=\int_0^1 x^h dx=\left.\frac{x^{h+1}}{h+1}\right|_0^1=\frac{1}{h+1}.$$

($\Longleftarrow$) For the other direction, consider the family $\mathcal{A}=\operatorname{span}\{x^h\mid x\in[0,1],h\in\mathbb{N}_0\}$. By hypothesis and linearity, we have that for all $f\in\mathcal{A}$.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\})=\int_0^1 f(x)dx.$$

Notice that $\mathcal{A}$ is also a subalgebra of $C([0,1])$ (that is, a vector subspace of $C([0,1])$ that is closed under multiplication of functions) and it is trivially closed under taking complex conjugation. Also, it separates points (given that the identity $f(x)=x$ is part of the family) and contains the constant functions. Thus, by the Stone-Weierstrass Theorem, we have that $\mathcal{A}$ is dense in $C([0,1])$. Let $f\in C([0,1])$ and let $(f_k)_k\subseteq\mathcal{A}$, such that $f_n\to f$ as $n\to\infty$.

Notice that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\})-\int_0^1 f(x)dx\right|\\
&\leq\left|\frac{1}{N}\sum_{n=1}^{N}(f(\{x_n\}-f_k(\{x_n\})\right|+\left|\int_0^1(f_k(x)-f(x))\,dx\right|+\left|\frac{1}{N}\sum_{n=1}^{N}f_k(\{x_n\})-\int_0^1 f_k(x)dx\right|\\
&\leq 2\|f-f_k\|_\infty+\left|\frac{1}{N}\sum_{n=1}^{N}f_k(\{x_n\})-\int_0^1 f_k(x)dx\right|.
\end{aligned}$$

Taking limits, we infer that

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)dx\right| \leq 2||f-f_k||_\infty,$$

and taking $k\to\infty$,

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)dx\right| \leq 0.$$

Thus, we conclude that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx,$$

which implies that $\{x_n\}$ is equidistributed by Weyl's criterion.

P3. (a) Prove that the sequence $(\log n)_{n\in\mathbb{N}}$ is not uniformly distributed mod 1.
Hint: Use Euler's summation formula: If $N\in\mathbb{N}$ and $F\in C^1([1,N])$, then

$$\sum_{n=1}^{N} F(n) = \int_1^N F(t)dt + \frac{F(1)+F(N)}{2} + \int_1^N \Big(\{t\}-\frac{1}{2}\Big)F'(t)dt,$$

where F' is the derivate of F.

We will use the Weyl's equidistribution criterion. Let $F(n)=e(\log n)$. Using the hint, we have that

$$\frac{1}{N}\sum_{n=1}^{N} e(\log n) = \frac{1}{N}\int_1^N e(\log t)dt + \frac{1+e(\log N)}{2N} + \frac{2\pi i}{N}\int_1^N \Big(\{t\}-\frac{1}{2}\Big)\frac{e(\log t)}{t}dt.$$

The second term of the sum is $O(N^{-1})$ and, hence, goes to 0 as $N\to\infty$. For the third term, we have that

$$\left|\frac{2\pi i}{N}\int_1^N \Big(\{t\}-\frac{1}{2}\Big)\frac{e(\log t)}{t}dt\right| \leq \frac{2\pi}{N}\int_1^N \left|\{t\}-\frac{1}{2}\right|\frac{1}{t} \leq \frac{\pi\log\ N}{N},$$

which goes to 0 as $N\to\infty$. Finally, for the first term, we have that

$$\frac{1}{N}\int_1^N e(\log t)dt = \frac{1}{N}\int_0^{\log\ N} e^{(2\pi i+1)u}du = \frac{e^{(2\pi i+1)\log\ N}-1}{N(2\pi i+1)} = \frac{e(\log N)}{(2\pi i+1)} - \frac{1}{N(2\pi i+1)}.$$

Notice that this sequence does not converge as $e(\log N)$ does not converge. Therefore, $(\log n)_n$ is not uniformly equidistributed mod 1.

(b) Optional: We say that a sequence $x_n\in[0,1)$ is uniformly distributed with respect to logarithmic averages[1] if for every $0\leq a\leq b\leq 1$, we have

$$\lim_{N\to+\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{\mathbb{1}_{[a,b)}(x_n)}{n} = (b-a).$$

Prove that if a sequence is uniformly distributed in the classical sense, then it is uniformly distributed with respect to logarithmic averages.
Hint: Use summation by parts.

[1]Compare this to the classical notion of uniform distribution using the Cesàro averages $\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{[a,b)}(x_n)$.

Fix an arbitrary interval $[a,b)$ in $[0,1)$. We let $S_M = \sum_{n=1}^{M} \mathbb{1}_{[a,b)}(x_n)$ (we define $S_0 = 0$) and our assumption is that $S_M/M \to (b-a)$ as $M \to +\infty$. We write

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{\mathbb{1}_{[a,b)}(x_n)}{n} = \frac{1}{\log N}\sum_{n=1}^{N}\frac{S_n - S_{n-1}}{n} = \frac{1}{\log N}\sum_{n=1}^{N-1} S_n\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{S_N}{N\log N}. \quad (1)$$

Using our hypothesis, we have that the second term tends to zero as $N \to +\infty$. To handle the first sum, fix $\varepsilon > 0$ and find $N_0 \in \mathbb{N}$ such that $\left|\frac{S_n}{n} - (b-a)\right| < \varepsilon$ for all $n \geq N_0$. Therefore, we have

$$\left|\frac{S_n}{n(n+1)} - \frac{b-a}{n+1}\right| \leq \frac{\varepsilon}{n+1}$$

for all $n \geq N_0$. Hence,

$$\left|\sum_{n=N_0}^{N}\frac{S_n}{n(n+1)} - \varepsilon\sum_{n=N_0}^{N}\frac{b-a}{n+1}\right| \leq \varepsilon\sum_{n=N_0}^{N}\frac{1}{n+1}$$

For the contribution of the terms with $n \leq N_0 - 1$, we use a crude bound

$$\left|\sum_{n=1}^{N_0-1}\frac{S_n}{n(n+1)} - \sum_{n=1}^{N_0-1}\frac{b-a}{n+1}\right| \leq \sum_{n=1}^{N_0-1}\left|\frac{S_n}{n(n+1)} - \frac{b-a}{n+1}\right| \leq 100N_0.$$

Therefore,

$$\left|\frac{1}{\log N}\sum_{n=1}^{N}\frac{S_n}{n(n+1)} - \frac{1}{\log N}\sum_{n=1}^{N}\frac{b-a}{n+1}\right| \leq \frac{1}{\log N}\left(100N_0 + \sum_{n=N_0}^{N}\frac{1}{n+1}\right).$$

Using the fact that

$$\lim_{N\to+\infty}\frac{1}{\log N}\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) = 1$$

we conclude that

$$\limsup_{N\to+\infty}\left|\frac{1}{\log N}\sum_{n=1}^{N}\frac{S_n}{n(n+1)} - (b-a)\right| \leq \varepsilon.$$

Since ε was arbitrary, we deduce that

$$\lim_{N\to+\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{S_n}{n(n+1)} = b-a.$$

Plugging this in (1), we deduce that

$$\lim_{N\to+\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{\mathbb{1}_{[a,b)}(x_n)}{n} = b-a.$$

Since a,b were arbitrary, we reach our conclusion.

(c) Prove that the sequence $\log n \pmod 1$ is uniformly distributed with respect to logarithmic averages. Conclude, in particular, that the sequence $\{\log n\}$ is dense in $[0,1]$ (you may assume without proof in this exercise that Weyl's criterion holds for logarithmic averages).

It suffices to show that for any $m \in \mathbb{Z}$ not equal to zero, we have

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{e(m\{\log n\})}{n} = 0. \tag{2}$$

This follows using the same arguments used in the proof of Weyl's criterion by replacing Cesàro averages with logarithmic averages.

We apply Euler's summation formula to the function $F(t) = \frac{e(m \log t)}{t}$ to deduce that

$$\sum_{n=1}^{N} \frac{e(m \log n)}{n} = \int_1^N \frac{e(m \log t)}{t} dt + \frac{1 + \frac{e(m \log N)}{N}}{2} + \int_1^N \left(\{t\} - \frac{1}{2}\right) \frac{(2\pi i m - 1) e(m \log t)}{t^2} dt. \tag{3}$$

It suffices to show that each term in the sum is $o(\log N)$. We handle the second term first, since it is simpler. We have that

$$\left| \frac{1}{\log N} \cdot \frac{N + e(m \log n)}{2N} \right| \leq \frac{N+1}{2N \log N} \to 0 \text{ as } N \to +\infty.$$

We now work with the third term in (3). We have

$$\frac{1}{\log N} \left| \int_1^N \left(\{t\} - \frac{1}{2}\right) \frac{(2\pi i m - 1) e(m \log t)}{t^2} dt \right| \leq \frac{1}{\log N} \int_1^N \left| \frac{(2\pi i m - 1) e(m \log t)}{t^2} \left(\{t\} - \frac{1}{2}\right) \right| dt \leq$$
$$\frac{|2\pi i m - 1|}{\log N} \int_1^N \frac{dt}{t^2} \left(\{t\} + \frac{1}{2}\right) < \frac{3|2\pi i m - 1|}{2 \log N},$$

using the fact that $\{t\} \leq 1$ for all $t \in \mathbb{R}$ and that $\int_1^N \frac{dt}{t^2} < 1$ for all $N \in \mathbb{N}$. Thus, the third term is also $o(\log N)$ (in fact, the corresponding integral is bounded). Finally, we estimate the first term:

$$\frac{1}{\log N} \left| \int_1^N \frac{e(m \log t)}{t} dt \right|$$

Here, we calculate

$$\int_1^N \frac{e(m \log t)}{t} dt = \int_1^N \left(\frac{e(m \log t)}{2\pi i m} \right)' dt = \frac{e(m \log t)}{2\pi i m} \Big|_1^N = \frac{e(m \log N) - 1}{2\pi i m}$$

and, thus,

$$\frac{1}{\log N} \left| \int_1^N \frac{e(m \log t)}{t} dt \right| = \frac{1}{\log N} \frac{|e(m \log N) - 1|}{2|\pi i m|} \leq \frac{1}{\pi m \log N}.$$

Combining everything together, we conclude that

$$\lim_{N \to +\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{e(m \log n)}{n} = 0$$

and thus (3) holds.

The fact that $\{\log n\}$ is dense is now obvious, since for any interval (a, b) there exist infinitely many $n \in \mathbb{N}$ such that $\mathbb{1}_{[a,b)}(\{\log n\}) = 1$.

P4. Let (y_n) be a sequence of distinct integers. For every $m \in \mathbb{N}$ define the sequence of functions

$$S_{m,N}(x) = \frac{1}{N}\sum_{n=1}^{N} e(my_n x).$$

(a) Prove that $S_{m,N}(x) \to 0$ as $N \to +\infty$ for almost all $x \in [0,1]$.
Hint: Compute $\|S_{m,N^2}(x)\|_{L^2([0,1))}$ and show that $S_{m,N^2}(x) \to 0$ for almost all (with respect to the Lebesgue measure) $x \in [0,1]$.

For all $N \in \mathbb{N}$, we have

$$\int_0^1 |S_{m,N}(x)|^2\,dx = \frac{1}{N^2}\int_0^1 \sum_{1\le n_1,n_2\le N} e(my_{n_1}x - myn_2x)dx = \frac{1}{N^2}\sum_{1\le n_1,n_2\le N} \mathbb{1}_{n_1=n_2} = \frac{1}{N}.$$

where we used the fact that

$$\int_0^1 e(kx) = \begin{cases} 1, & k=0 \\ 0, & \text{otherwise} \end{cases}.$$

Thus, we conclude that

$$\sum_{N=1}^{+\infty}\int_0^1 \left|S_{m,N^2}(x)\right|^2 dx = \sum_{n=1}^{+\infty}\frac{1}{N^2} < +\infty$$

and the monotone convergence theorem implies that

$$\int_0^1 \sum_{N=1}^{+\infty} \left|S_{m,N^2}(x)\right|^2 dx = \sum_{N=1}^{+\infty}\int_0^1 \left|S_{m,N^2}(x)\right|^2 dx < +\infty.$$

We conclude that for almost all $x \in [0,1]$ the series $\sum_{N=1}^{+\infty}\left|S_{m,N^2}(x)\right|^2$ is finite. Hence, we have $\lim_{N\to+\infty} S_{m,N^2}(x) \to 0$ for almost all $x \in [0,1]$.
Now, given a positive integer N, we find an integer k such that $k^2 \le N < (k+1)^2$. We have

$$|S_{m,N}(x)| \le \left|\frac{1}{N}\sum_{n=1}^{k^2} e(my_nx)\right| + \left|\frac{1}{N}\sum_{k^2<n\le N} e(my_nx)\right| \le \left|\frac{1}{k^2}\sum_{n=1}^{k^2} e(my_nx)\right| + \frac{2k}{N} = \left|S_{m,k^2}(x)\right| + \frac{2}{k}.$$

We conclude that $\lim\limits_{N\to+\infty} S_{m,N}(x) = 0$ for almost all $x \in [0,1]$.

(b) Prove that for almost all $a \in [0,1]$, the sequence $(y_n a)_{n\in\mathbb{N}}$ is uniformly distributed mod 1. As an application, conclude that for almost all real numbers the sequence $\{b^n x\}$ is uniformly distributed mod 1 for all $b \in \mathbb{N}$ with $b \ge 2$.

For every $m \in \mathbb{N}$, define the set

$$A_m = \{x \in [0,1] \colon |S_{m,N}(x)| \not\to 0\}.$$

We have shown that $\mu(A_m) = 0$ for all $m \in \mathbb{N}$, and thus $\mu(\bigcup_{m\in\mathbb{N}} A_m) = 0$. Therefore, for every

$x \in [0,1] \setminus \left(\bigcup_{m\in\mathbb{N}} A_m\right)$ we have that

$$\operatorname{lím} \frac{1}{N}\sum_{n=1}^{N} e(my_n x) = 0$$

and hence the sequence $(y_n x)$ is uniformly distributed mod 1 by Weyl's criterion. Since the set $[0,1] \setminus \left(\bigcup_{m\in\mathbb{N}} A_m\right)$ has measure 1, this establishes our claim.

First, our previous claims imply that, for a fixed $b \geq 2$, almost all real numbers in $[0,1]$ satisfy that $\{b^n x\}$ is uniformly distributed. More precisely, let A_b be the set of all $x \in [0,1]$ such that the sequence $\{b^n x\}$ is not uniformly distributed. Then, $\mu(A_b) = 0$ and, thus, $\mu(\bigcup_{b\geq 2} A_b) = 0$. For every $x \in [0,1] \setminus \bigcup_{b\geq 2} A_q$, we have that $\{b^n x\}$ is uniformly distributed for all integers $b \geq 2$.

To show the claim for almost all real numbers, we observe that x satisfies the property in the statement if and only if $x - \lfloor x \rfloor$ satisfies the same property. Thus, for any $k \in \mathbb{Z}$, we have

$$\mu\left(\{x \in [k, k+1] \colon \text{ there exists } b \text{ such that } \{b^n x\} \text{ is not uniformly distributed}\}\right) = 0.$$

The conclusion follows from the fact that a countable union of zero measure sets has measure zero.